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SOLUTION OF BESSEL DIFFERENTIAL EQUATION OF ORDER ZERO BY USING DIFFERENT METHODS IN CRITICAL STUDY

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ABSTRACT

Two different methods for solving Bessel differential equation of order zero with comparison to the resulting solutions was carried out in this work.

Keywords: *Bessel differential equation, Laplace transforms , power series.*

I. INTRODUCTION

The Bessel differential equation is the linear second-order ordinary differential equation, it considered one of the most important ordinary differential equations due it is wide applications such as heat transfer, vibrations, stress analysis and fluid mechanics. The solutions to this equation gives the Bessel functions (of first and second kinds) usually denoted by $J_n(x)$ and $Y_n(x)$ respectively they are firstly defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel. Bessel functions are associated with a wide range of problems in important areas of mathematical physics especially in problems of wave propagation and signal processing , see [1]. There are different methods to solve Bessel differential equation, in this article we used the Laplace transform method that named after mathematician and astronomer Pierre-Simon Laplace which is a powerful integral transform methods to solve linear differential equations with given initial conditions ,see [2], and the power series method which is very common method in solving differential equations, see [3], then we compared and discussed the resulting solutions obtained from each method in critical discussion.

Definition1: Bessel differential equation of order n is second order ordinary differential equation of the form :

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + \left(x^2 - n^2\right) y = 0 \quad , \quad n \geq 0 \tag{1}$$

Definition2: Bessel functions $J_n(x)$ of the first kind of order n are the solutions of Bessel's differential equation (1) which are finite at $x = 0$, for integer or positive n, and diverge when x approaches zero for negative non-integer n.

By using the series expansion of the Bessel functions of the first kind, $J_n(x)$ around $x = 0$, we can define $J_n(x)$ in term of is the gamma function as :

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} \tag{2}$$

Since $\Gamma(m+n+1) = (m+n)!$, then (2) can be written as :

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n} \tag{3}$$

see[5] . The formula

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$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n} \tag{4}$$

is obtained by replacing n in (2) by $-n$. If n is positive integer, for $m = 0,1,2,3,\cdots n-1$ the value $m-n-+1 \leq 0$ and then all the coefficients in (4) are not defined since gamma function is not defined at zero and for negative integer, see[5].

$J_n(x)$ and $J_{-n}(x)$ are linearly independent only when n is not an integer.

The Bessel functions of the second kind, $Y_n(x)$ are solutions of the Bessel differential equation that have a singularity at $x = 0$, and are multivalued.

When n is not an integer $Y_n(x)$ defined in terms of $J_n(x)$ and $J_{-n}(x)$ as:

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}}{\sin(n\pi)} \tag{5}$$

General solution of Bessel differential equation of order *n*

If n is not an integer, the general solution of Bessel differential equation of order n (1) is of the form:

$y = C_1 J_n(x) + C_2 Y_n(x)$ where C_1 and C_2 are given constants.

The values for the Bessel functions can be found in most collections of mathematical tables.

Definition3: Bessel differential equation of order zero is the second order ordinary differential equation of the form:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0 \tag{6}$$

Definition4. Let $F(t)$ be a given continuous function , for $t > 0$. Then the Laplace transform of $F(t)$ denoted by $L\left[F(t)\right]$ is defined by:

$$L\left[F(t)\right] = f(s) = \int_0^{\infty} F(t)\,e^{-st}\,dt \tag{7}$$

The Laplace transform of $F(t)$ is said to exist if the integral (7) converges for some value of s, otherwise it does not exist. For the sufficient conditions for existence of Laplace transform see [2].

Theorem 5: If $F(t)$ is given continuous function for $0 \leq t \leq N$ and of exponential order for $t > N$ while $F'(t)$ is sectionally continuous for $0 \leq t \leq N$, and if $L\left[F(t)\right] = f(s)$, then $L\left[F'(t)\right] = s\,f(s) - F(0)$.

Theorem6: If $F(t), F'(t), F''(t), \cdots, F^{(n)}(t)$ is given continuous functions for $0 \leq t \leq N$ and of exponential order for $t > N$ while $F^{(n)}(t)$ is sectionally continuous for $0 \leq t \leq N$, and if $L\left[F(t)\right] = f(s)$, then

$L\left[F^{(n)}(t)\right] = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^0 f^{(n)}(0)$.

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Definition7: If $L\left[F(t)\right] = f(s)$, then $F(t)$ is called an inverse Laplace transform of $f(s)$ and we write symbolically $F(t) = L^{-1}\left[f(s)\right]$ where L^{-1} is called the inverse Laplace transformation operator.

Theorem8: If $L^{-1}\left[f(s)\right] = F(t)$, then $L^{-1}\left[f^{(n)}(s)\right] = L^{-1}\left[\frac{d}{ds^n} f(s)\right] = (-1)^n t^n F(t)$

Definition9: The power series method for solving differential equation consist of substituting the power series

$$y = \sum_{0}^{\infty} c_m x^m \tag{8}$$

In the given differential equation, and then determine the coefficients $c_1, c_2 c_3, \cdots, c_n$ in order that the power series will satisfy the given differential equation.

Formulation of the problem

Consider the second order ordinary differential equation

$$x\frac{d^2 y}{dx^2} + \frac{dy}{dx} + 4xy = 0 \tag{9}$$

Which is the equivalent equation to the Bessel differential equation of order zero. Our considered problem is to solve equation (9) subject to the initial conditions $(i)\ y(0) = 3 \qquad (ii)\ y'(0) = 0$ by using two different methods which are power series and Laplace transform method.

Power series method:

Assume that (9) has solution of the form:

$$y = \sum_{0}^{\infty} a_r x^{c+r} \tag{10}$$

Where a and c are constants which must be determined, then when we substitute the solution $y = \sum_{0}^{\infty} a_r x^{c+r}$ and it is derivatives y' and y'' in (9) we get

$$x \sum_{0}^{\infty}(c+r)(c+r-1) a_r x^{c+r-2} + \sum_{0}^{\infty}(c+r)\ a_r x^{c+r-1} + 4x \sum_{0}^{\infty} a_r x^{c+r} = 0 \tag{11}$$

When we equal the coefficients of x^{c-1} in the series we get that $c(c+1)a_0 + c a_0 = 0$ since $a \neq 0$ then $c = 0$, when we apply the conditions (i) and (ii) we get that

$$a_0 = 3 \ \ and \ \ a_1 = 0 \tag{12}$$

When we equal the coefficients of x^{r-1} in the series we get that $r(r-1)a_r + r a_r + a_{r-2} = 0$ that is $r^2 a_r = - a_{r-2}$ then the recurrence relation is

$a_r = \frac{-4}{r^2} a_{r-2}$, $r \geq 2$, then we conclude that

$$a_1 = a_3 = a_5 = \cdots = 0 \tag{13}$$

and

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$$a_2 = -a_0 , a_4 = \frac{-1}{4} a_2 , \quad a_6 = \frac{-1}{9} a_4 , a_8 = \frac{-1}{16} a_6 , \cdots \tag{14}$$

From (12) , (13)and (14) we conclude that the series solution of (9) is of the form:

$$y = 3 - 3x^2 + \frac{3}{4} x^4 - \frac{1}{12} x^6 + \frac{1}{192} x^8 + \cdots + a_r x^r + \cdots \tag{15}$$

which is the function that satisfy the equation and the conditions.

Laplace transforms method:

The criteria of this method is to take Laplace transform of equation (9),solving the resulting equation and finally take Laplace inverse to get the required solution.

The Laplace transform of the equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4xy = 0$ or $x\left(y''+4y\right) + y' = 0$ gives

$$-\frac{d}{dx} L[\ y'' + 4y] + L[y'] = L[0]$$

$$-\frac{d}{ds}\left(s^2 L[y] - s\,y(0) - y'(0) + 4L[y] \right) + \left(s\,L(y) - y(0)\right) = 0, \text{ by using (i) \&(ii)}$$

$$= -\frac{d}{ds}\left(s^2 L[y] - 3s + 4\ L[y] \right) + \left(s\ L[y] - 3 \right) = 0$$

$$= -\frac{d}{ds}\left((s^2+4)\,L[y] - 3s \right) + \left(s\ L[y] - 3 \right) = 0$$

$$= \left(s^2+4\right)\frac{d\,L[y]}{ds} = -s\,L[y] \quad \Rightarrow \quad \frac{d\,L[y]}{L[y]} = \frac{1}{2}\left(\frac{-2\,s\,ds}{\left(s^2+4\right)}\right)$$

$$\Rightarrow L[y] = \frac{c}{\sqrt{s^2+4}} = \frac{c}{s}\,\frac{1}{\sqrt{1+\frac{4}{s^2}}}$$

$$= \frac{c}{s}\left[1 - \frac{1}{2}\left(\frac{4}{s^2}\right) + \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{4}{s^2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{1}{s^2}\right)^3 + \cdots\right]$$

$$= \frac{3}{s} - \frac{6}{s^2} + \frac{18}{s^5} - \frac{20}{s^7} + \cdots$$

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$$= 3\left(\frac{1}{s}\right) - \frac{3}{2!}\left(\frac{2}{s^2}\right) + \frac{18}{4!}\left(\frac{4!}{s^5}\right) - \frac{20}{6\,!}\left(\frac{6!}{s^7}\right) \cdots$$

$$y = L^{-1}[y] = L^{-1}\left[\ 3\left(\frac{1}{s}\right) - 3\left(\frac{2}{s^2}\right) + 3\left(\frac{6}{s^5}\right) - \cdots\right]$$

$$= 3 - 3x^2 + \frac{3}{4}x^4 - \frac{1}{12}x^6 + \frac{1}{192}x^8 + \cdots \tag{16}$$

See the table of Laplace transforms in [6]. The solution (16) is the same result that obtained when we used power series method.

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